

# Poisson Structure of the Boundary Gravitons in 3D Gravity with Negative $\Lambda$ .

Cédric Troessaert<sup>a</sup>

*Centro de Estudios Científicos (CECs)*

*Arturo Prat 514, Valdivia, Chile*

*troessaert@cecs.cl*

**ABSTRACT.** We use the hamiltonian formalism to study the asymptotic structure of 3 dimensional gravity with a negative cosmological constant. We start by defining very general fall-off conditions for the canonical variables and study the implied Poisson structure of the boundary gravitons. From the allowed differentiable gauge transformations, we can extract all the possible boundary conditions on the lagrange multipliers and the associated boundary hamiltonians. In the last section, we use this general framework to describe some of the previously known boundary conditions.

<sup>a</sup>Laurent Houart postdoctoral fellow.

# Contents

|          |  |           |
|----------|--|-----------|
| <b>1</b> | <b>Introduction</b>                                | <b>3</b>  |
| <b>2</b> | <b>Asymptotic structure</b>                        | <b>4</b>  |
| 2.1      | Differentiable gauge transformations . . . . .     | 5         |
| 2.2      | Boundary gravitons . . . . .                       | 9         |
| 2.3      | Dirac bracket for the boundary gravitons . . . . . | 10        |
| <b>3</b> | <b>Boundary hamiltonian</b>                        | <b>12</b> |
| <b>4</b> | <b>Some examples of boundary conditions</b>        | <b>13</b> |
| 4.1      | Conformal . . . . .                                | 13        |
| 4.2      | Brown-Henneaux . . . . .                           | 15        |
| 4.3      | Chiral . . . . .                                   | 18        |
| 4.4      | Constrained Chiral . . . . .                       | 20        |
| <b>A</b> | <b>Radial decomposition</b>                        | <b>22</b> |
| <b>B</b> | <b>Gauge transformations</b>                       | <b>23</b> |
| <b>C</b> | <b>Fefferman-Graham gauge fixing</b>               | <b>24</b> |

# 1 Introduction

Since its introduction, Einstein's theory in three dimensions has been a very useful toy model to study properties of gravitational theories. Even if it lacks some features compared to its higher dimensional versions, like gravitational waves, it still possesses dynamical objects [1] and black-holes [2, 3].

This theory is particularly interesting in the context of AdS/CFT. In their seminal work [4], Brown and Henneaux showed that the algebra of the conserved charges of asymptotically  $AdS_3$  space-times is given by two copies of the Virasoro algebra with non-zero central charge. This lead to many interesting results, for instance: Strominger was able to reproduce the Bekenstein-Hawking entropy of the BTZ black-holes using the Cardy formula [5]. Since then, this framework has been extended: either by relaxing the original asymptotic conditions of Brown-Henneaux [6, 7] or introducing new asymptotics with different boundary dynamics [8, 9]. We now have a few different sets of boundary conditions available but it is reasonable to say that a lot more possibilities should exist.

Using the Chern-Simons description of 3D gravity, one can solve the constraints and obtain the reduced theory describing the dynamics of the boundary gravitons. For Brown-Henneaux boundary conditions, this procedure leads to a Liouville theory on the boundary [10, 11, 12, 13]. On the other hand, for chiral boundary conditions, one obtains a chiral Liouville theory on the boundary [8]. All these results rely heavily on the fact that one can solve the constraints and are difficult to generalize in different contexts.

In this work, we use the hamiltonian framework to provide a unified description of the previously introduced boundary conditions. The idea is to start with very general asymptotic fall-off conditions and use the results obtained in [14]. In the process, we will build a description of the reduced theory living on the boundary at infinity without explicitly solving the constraints.

In the first section, we study the asymptotic structure of 3D gravity with a negative cosmological constant. We introduce our asymptotic fall-off conditions and study the structure of the reduced phase-space. More precisely, we build quantities parametrizing the boundary gravitons and compute the induced poisson structure.

In the second section, we describe all possible boundary conditions on the lagrange multipliers. These boundary conditions are responsible for the dynamical part of the theory. In particular, they are in one to one correspondence with the induced hamiltonian on the phase-space of the boundary gravitons.

In the last section, we use our formalism to describe some of the boundary conditions previously obtained in the literature. We study both the conformally symmetric boundary conditions [7, 4] and the chiral boundary conditions [9, 8]

In [15], the authors conjectured that all the previously introduced asymptotic conditions for 3D gravity are dual to Polyakov 2D gravity with different gauge choices for the metric. It would be interesting to see how their approach can be extended to the most general asymptotic conditions introduced here.

In this paper, we use the notation  $O(r^n)$  to describe functions with the following behavior in the limit  $r \rightarrow \infty$ :

$$f(r, x^A) = O(r^n) \quad \Rightarrow \quad \lim_{r \rightarrow \infty} \frac{f}{r^n} = \bar{f}(x^A). \quad (1.1)$$

We will also ask for a compatible behavior with as many partial derivatives as needed:

$$f(r, x^A) = O(r^n) \quad \Rightarrow \quad \partial_r^k f(r, x^A) = O(r^{n-k}) \quad \text{and} \quad \partial_A^k f = O(r^n). \quad (1.2)$$

## 2 Asymptotic structure

The bulk hamiltonian action for gravity in 3 dimension is given by:

$$S[N, N^i, g_{ij}, \pi^{ij}] = \frac{1}{16\pi G} \int dt \int_{\Sigma} d^2x \{ \pi^{ij} \partial_t g_{ij} - N \mathcal{R} - N^i \mathcal{R}_i \}, \quad (2.1)$$

$$\mathcal{R} = -\sqrt{g} \left[ R - 2\Lambda + \frac{1}{g} (\pi^2 - \pi^{ij} \pi_{ij}) \right], \quad (2.2)$$

$$\mathcal{R}_i = -2\nabla_j \pi_i^j, \quad (2.3)$$

where  $g_{ij}$  is a 2 dimensional metric and  $\pi^{ij}$  is a density. In order to apply the formalism of [14], we need boundary conditions on the dynamical variables  $(g_{ij}, \pi^{ij})$ . As, we want to study the asymptotic structure, we need fall-off conditions in order to have generators given by finite quantities. The most common choice is the one used in [4] but there have been other propositions [6, 8, 7, 9]. Following the results of [14], we expect these boundary conditions to share the same reduced phase-space, the differences being in the choice of the Hamiltonian.

We will start with general fall-off conditions on the phase-space containing all of the previously proposed boundary conditions. The analysis of the boundary conditions on the lagrange multipliers will be postponed to the study of the hamiltonian generators starting in section 4.1. We will consider the following asymptotic behavior:

$$g_{rr} = \frac{l^2}{r^2} + O(r^{-4}), \quad g_{r\phi} = O(r^{-1}), \quad g_{\phi\phi} = r^2 \bar{\gamma}(t, \phi) + O(1), \quad (2.4)$$

$$\pi^{rr} = O(r), \quad \pi^{r\phi} = O(r^{-2}), \quad \pi^{\phi\phi} = O(r^{-5}), \quad (2.5)$$

where  $\Lambda = -\frac{1}{l^2}$  and  $\bar{\gamma}$  is a dynamical field which is always positive. In [4], the authors showed that such fall-off conditions are not enough for a hamiltonian analysis of the

problem. We also have to impose the constraints asymptotically:

$$\mathcal{R} = O(r^{-n}), \quad \mathcal{R}_i = O(r^{-n}) \quad \forall n \in \mathbb{N}. \quad (2.6)$$

With this set of fall-off conditions, the bulk part of the action (2.1) is finite whenever the lagrange multipliers satisfy

$$\exists m \in \mathbb{N} \quad \text{s.t.} \quad N = O(r^m), \quad N^i = O(r^m). \quad (2.7)$$

The additional conditions on the constraints (2.6) have some useful consequences. In particular, we have:

$$\pi^{rr} = \frac{r}{2l} P(t, \phi) + O(r^{-1}), \quad (2.8)$$

and

$$\partial_r \left( r^2 \left( K + \frac{1}{l} \right) \right) = O(r^{-3}), \quad (2.9)$$

where  $K$  is the trace of the extrinsic curvature of the circles  $r$  equals constant (see appendix A).

## 2.1 Differentiable gauge transformations

Gauge-like transformations are given by:

$$g_{ij} = \frac{\delta(\xi \mathcal{R} + \xi^k \mathcal{R}_k)}{\delta \pi^{ij}}, \quad \pi^{ij} = -\frac{\delta(\xi \mathcal{R} + \xi^k \mathcal{R}_k)}{\delta g_{ij}}, \quad (2.10)$$

where the gauge parameters  $\xi, \xi^i$  can depend on the fields. We will restrict our analysis to gauge parameters with the following asymptotic behavior:

$$\xi = O(r), \quad \xi^r = O(r), \quad \xi^\phi = O(1). \quad (2.11)$$

In this case, using (2.6), the explicit form of the gauge-like transformations is worked out to be:

$$\begin{aligned} \delta_\xi \pi^{ij} &= -\sqrt{g} \xi \Lambda g^{ij} + \sqrt{g} (\nabla^i \nabla^j \xi - g^{ij} \nabla^k \nabla_k \xi) \\ &\quad - 2 \frac{\xi}{\sqrt{g}} (\pi^{ik} \pi_k^j - \pi \pi^{ij}) - \frac{\xi}{\sqrt{g}} \frac{g^{ij}}{2} (\pi^2 - \pi^{kl} \pi_{kl}), \\ &\quad + \partial_k (\xi^k \pi^{ij}) - \partial_k \xi^i \pi^{kj} - \partial_k \xi^j \pi^{ki} + O(r^{-n}), \end{aligned} \quad (2.12)$$

$$\delta_\xi g_{ij} = 2 \frac{\xi}{\sqrt{g}} (\pi_{ij} - \pi g_{ij}) + \xi^k \partial_k g_{ij} + \partial_i \xi^k g_{kj} + \partial_j \xi^k g_{ki} + O(r^{-n}), \quad (2.13)$$

for all  $n \in \mathbb{R}$ .

A differentiable gauge transformation is a gauge-like transformation  $\delta_\xi$  for which we can associate a differentiable generator. This requires two conditions to be met: the transformation  $\delta_\xi$  preserves the boundary conditions and the generator  $\Gamma_\xi$  satisfies

$$\delta\Gamma_\xi = \int_\Sigma d^2x \left( \frac{\delta\Gamma_\xi}{\delta g_{ij}} \delta g_{ij} + \frac{\delta\Gamma_\xi}{\delta \pi^{ij}} \delta \pi^{ij} \right). \quad (2.14)$$

To compute the set of differentiable gauge transformations, we will start by computing the set of gauge-like transformations (2.12)-(2.13) preserving the boundary conditions.

The variations of the constraints under a gauge-like transformation are given by:

$$\delta_\xi \mathcal{R} = \partial_i(\xi^i \mathcal{R}) - \partial_i(\xi g^{ij} \mathcal{R}_j) - \partial_i \xi g^{ij} \mathcal{R}_j + O(r^{-n}), \quad \forall n \in \mathbb{R}, \quad (2.15)$$

$$\delta_\xi \mathcal{R}_k = \partial_k \xi \mathcal{R} + \partial_i(\xi^i \mathcal{R}_k) + \partial_k \xi^i \mathcal{R}_i + O(r^{-n}), \quad \forall n \in \mathbb{R}. \quad (2.16)$$

We see that any transformation of the form (2.11) will preserve the fall-off conditions on the constraints. Computing the variation of the metric and using the fall-off conditions, we obtain:

$$\delta_\xi g_{rr} = 2 \frac{l^2}{r^2} (\partial_r \xi^r - \frac{1}{r} \xi^r) + 2 \partial_r \xi^\phi g_{r\phi} + O(r^{-4}), \quad (2.17)$$

$$\delta_\xi g_{r\phi} = r^2 \sigma \partial_r \xi^\phi + O(r^{-1}), \quad (2.18)$$

$$\delta_\xi g_{\phi\phi} = -2\xi \sqrt{g} \pi^{rr} + \xi^r \partial_r g_{\phi\phi} + \xi^\phi \partial_\phi g_{\phi\phi} + 2 \partial_\phi \xi^\phi g_{\phi\phi} + O(1). \quad (2.19)$$

The preservation of the fall-off conditions for  $g_{rr}$  and  $g_{r\phi}$  leads to

$$\xi^r = r\psi + O(r^{-1}), \quad \xi^\phi = Y + O(r^{-2}), \quad (2.20)$$

where  $\psi$  and  $Y$  are arbitrary functions independent of  $r$ . Taking this into account and using the spatial 1+1 decomposition of the metric described in appendix A, the variations of the momenta become

$$\delta_\xi \pi^{rr} = -\sqrt{\sigma} \frac{r^3}{l^3} (\partial_r \xi - \frac{1}{r} \xi) + O(r), \quad (2.21)$$

$$\delta_\xi \pi^{r\phi} = \frac{1}{l\sqrt{\sigma}} (\partial_\phi + \frac{r}{l^2} \lambda_\phi) (\partial_r \xi - \frac{1}{r} \xi) + O(r^{-2}), \quad (2.22)$$

$$\begin{aligned} \delta_\xi \pi^{\phi\phi} = & -\frac{1}{\sqrt{g}} \left( \partial_r^2 \xi - \frac{\lambda^2}{l^2} \xi - \frac{\partial_r \lambda}{\lambda} \partial_r \xi + \frac{r}{l^2} \lambda_\phi \lambda^\phi (\partial_r \xi - \frac{1}{r} \xi) \right. \\ & \left. - (3 \frac{\lambda^\phi}{r} + \partial_r \lambda^\phi) \partial_\phi \xi \right) + O(r^{-5}) \end{aligned} \quad (2.23)$$

where  $\lambda_\phi = O(r^{-1})$  and  $\lambda = \frac{l}{r} + O(r^{-3})$ . The preservation of the fall-off conditions for  $\pi^{rr}$  and  $\pi^{r\phi}$  then imply

$$\xi = rf + \kappa, \quad \kappa = O(r^{-1}), \quad (2.24)$$

where  $f$  is another arbitrary function independent of  $r$ . Using this and the asymptotic form of  $\pi^{rr}$  given in (2.8), the variation of  $g_{\phi\phi}$  automatically preserves the fall-off condition (2.4).

The only condition we still need to check is the preservation of  $\pi^{\phi\phi} = O(r^{-5})$ . Using the expansion  $\lambda = \frac{l}{r} + \tilde{\lambda}$  with  $\tilde{\lambda} = O(r^{-3})$ , we can simplify the variation (2.23) to:

$$\delta_\xi \pi^{\phi\phi} = -\frac{r^{-2}}{\sqrt{g}} \partial_r \left[ r^2 \left( \partial_r \kappa - \frac{1}{r} \kappa - \frac{\tilde{\lambda}}{l} r f - \lambda^\phi r \partial_\phi f \right) \right] + O(r^{-5}). \quad (2.25)$$

Any possible transformation satisfying this condition will also preserve the more constrained form of  $\pi^{rr}$  given in equation (2.8). Computing the variation of  $\pi^{rr}$  taking the gap into account leads to:

$$\delta_\xi \pi^{rr} = \frac{\gamma}{\lambda l} \left( \partial_r \kappa - \frac{1}{r} \kappa - \frac{\tilde{\lambda}}{l} r f - \lambda^\phi r \partial_\phi f - l \left( K + \frac{1}{l} \right) f \right) + \omega r + O(r^{-1}), \quad (2.26)$$

where  $\omega$  is a function independent of  $r$  encoding part of the variation of  $P$ . In order to preserve the asymptotic form of  $\pi^{rr}$ , the function  $\kappa$  must be of the form

$$\kappa = -\frac{l^2}{2r} \chi - r \int_r^\infty dr' j(r') + O(r^{-3}), \quad (2.27)$$

$$j = \frac{\tilde{\lambda}}{l} f + \lambda^\phi \partial_\phi f + \frac{l}{r} \left( K + \frac{1}{l} \right) f = O(r^{-3}), \quad (2.28)$$

where  $\chi$  is an arbitrary function independent of  $r$ . Combining equation (2.26) with (2.9), we see that such a  $\kappa$  induces a variation (2.25) that automatically preserves the fall-off of  $\pi^{\phi\phi}$ . We have shown the following:

**Theorem 2.1.** *The set of gauge-like transformations preserving the asymptotic conditions (2.4)-(2.6) is given by:*

$$\xi = r f - \frac{l^2}{2r} \chi - r \int_r^\infty dr' j(r') + O(r^{-3}), \quad (2.29)$$

$$\xi^r = r \psi + O(r^{-1}), \quad (2.30)$$

$$\xi^\phi = Y + O(r^{-2}), \quad (2.31)$$

where the function  $j$  is given in equation (2.28) and the four functions  $f, \chi, \psi$  and  $Y$  are independent of  $r$ .

The second condition for a gauge-like transformation to be differentiable is the existence of a differentiable generator. The bulk part of the generator of a gauge-like transformation is given by the smeared constraints:

$$\tilde{\Gamma}_\xi = \frac{1}{16\pi G} \int_\Sigma d^2x (\xi \mathcal{R} + \xi^i \mathcal{R}_i). \quad (2.32)$$

The boundary term coming from a general variation is then easily computed:

$$\begin{aligned} \delta \tilde{\Gamma}_\xi &= \int_\Sigma d^2x \left( \frac{\delta \tilde{\Gamma}_\xi}{\delta g_{ij}} \delta g_{ij} + \frac{\delta \tilde{\Gamma}_\xi}{\delta \pi^{ij}} \delta \pi^{ij} \right) \\ &+ \frac{1}{16\pi G} \oint_{\partial\Sigma} (d^1x)_k \left\{ -2\xi^i \delta \pi^k_i + \xi^k \pi^{ij} \delta g_{ij} - \xi \sqrt{g} (g^{ij} \delta \Gamma_{ij}^k - g^{ki} \delta \Gamma_{ji}^k) \right. \\ &\quad \left. + \partial_l \xi \sqrt{g} (g^{ki} g^{lj} - g^{kl} g^{ij}) \delta g_{ij} + \Theta^k(\mathcal{R}, \mathcal{R}_i) \right\}. \end{aligned} \quad (2.33)$$

The function  $\Theta$  is coming from the variation of the gauge parameters  $\xi, \xi^i$ : it is a local function of the constraints and their derivatives. In this case, as we have imposed the constraints asymptotically, it will always be zero. Inserting our fall-off conditions, the asymptotic form of the gauge parameters and evaluating at the boundary  $r \rightarrow \infty$ , the boundary term becomes:

$$-\frac{1}{16\pi G} \oint_{\partial\Sigma} d\phi \lim_{r \rightarrow \infty} \left\{ 2Y \delta \pi_\phi^r + l\psi \delta P + 2r^2 f \delta \left( \sqrt{\gamma} \left( K + \frac{1}{l} \right) \right) + 2 \left( r^2 \left( K + \frac{1}{l} \right) f + l\chi \right) \delta \sqrt{\gamma} \right\}. \quad (2.34)$$

Let's introduce the fields:

$$J(t, \phi) \equiv \frac{2}{l} \lim_{r \rightarrow \infty} \pi_\phi^r, \quad M(t, \phi) \equiv \frac{2\sqrt{\gamma}}{l} \lim_{r \rightarrow \infty} \left( r^2 \left( K + \frac{1}{l} \right) \right), \quad Q(t, \phi) \equiv 2\sqrt{\gamma}. \quad (2.35)$$

The boundary term (2.34) is integrable if and only if there exists a functional on the circle

$$\frac{l}{16\pi G} \oint_{\partial\Sigma} d\phi k_\xi(P, J, M, Q), \quad (2.36)$$

such that:

$$Y = \frac{\bar{\delta} k_\xi}{\delta J}, \quad \psi = \frac{\bar{\delta} k_\xi}{\delta P}, \quad (2.37)$$

$$f = \frac{\bar{\delta} k_\xi}{\delta M}, \quad \tilde{\chi} \equiv \chi + \frac{M}{Q} f = \frac{\bar{\delta} k_\xi}{\delta Q}, \quad (2.38)$$

where the Euler-Lagrange derivative  $\bar{\delta}$  is the one defined on the circle only:

$$\frac{\bar{\delta} k}{\delta M} = \sum_k (-\partial_\phi)^k \frac{\partial k}{\partial \partial_\phi^k M}. \quad (2.39)$$

If such a functional exists, the differentiable generator of the transformation is given by:

$$\Gamma_\xi = \frac{1}{16\pi G} \int_\Sigma d^2x (\xi \mathcal{R} + \xi^i \mathcal{R}_i) + \frac{l}{16\pi G} \oint_{\partial\Sigma} d\phi k_\xi(P, J, M, Q). \quad (2.40)$$

On the constraints, we obtain

$$\Gamma_\xi \approx \frac{l}{16\pi G} \oint_{\partial\Sigma} d\phi k_\xi(P, J, M, Q). \quad (2.41)$$

The transformations for which  $\Gamma_\xi \approx 0$  are called proper gauge transformations. They are the true gauge freedom of the system as they are generated by constraints and always commute with the differentiable Hamiltonian [14]. In the following, we will denote the parameters of proper gauge transformations by  $\eta$  and  $\eta^i$ .

The set differentiable gauge transformations form an algebra under the Poisson bracket for which the set of proper gauge transformations is an ideal. We have proved that



**Theorem 2.2.** *The quotient of the differentiable gauge transformation by the proper gauge transformations is parametrized by the functionals of  $P, J, M$  and  $Q$  defined on the circle:*

$$\frac{l}{16\pi G} \oint_{\partial\Sigma} d\phi k_\xi(P, J, M, Q). \quad (2.42)$$

The induced Poisson bracket on the quotient will be computed in section 2.3.

## 2.2 Boundary gravitons

We expect the quantities  $P, J, M$  and  $Q$  that we defined in the previous section to encode all the information about the boundary gravitons. More specifically, we expect them to be gauge invariant and to completely characterize the configuration up to proper gauge transformations.

The parameters of proper gauge transformations  $\Gamma_\eta$  have the following fall-off

$$\eta = O(r^{-3}), \quad \eta^r = O(r^{-1}), \quad \eta^\phi = O(r^{-2}). \quad (2.43)$$

We easily show that the associated transformations on the relevant canonical fields are given by:

$$\delta_\eta \pi^{rr} = O(r^{-1}), \quad \delta_\eta \pi^{r\phi} = O(r^{-4}), \quad \delta_\eta g_{\phi\phi} = O(1). \quad (2.44)$$

This means that  $P, J$  and  $Q$  are gauge invariant quantities. For  $M$ , we need the transformation law of  $K$  (see eq (B.4)). A straightforward computation gives

$$\delta_\eta K = O(r^{-4}), \quad (2.45)$$

which means that  $M$  is also gauge invariant.

In order to analyse the structure of the reduced phase-space, it is easier to fix the gauge. The simplest choice is the Fefferman-Graham gauge which is given by:

$$g_{rr} = \frac{l^2}{r^2}, \quad g_{r\phi} = 0, \quad \pi^{\phi\phi} = 0. \quad (2.46)$$

This gauge can always be reached by a proper gauge transformation (more details are given in appendix C). With the gauge fixed, the constraints simplify drastically:

$$\mathcal{R}_r = -2\frac{l^2}{r^2} \left( \partial_r \pi^{rr} - \frac{1}{r} \pi^{rr} + \partial_\phi \pi^{r\phi} \right), \quad (2.47)$$

$$\mathcal{R}_\phi = -2\gamma \frac{l}{r} \left( \frac{r}{l} \partial_r \pi^{r\phi} + \frac{2}{l} \pi^{r\phi} - 2\pi^{r\phi} \left( K + \frac{1}{l} \right) \right), \quad (2.48)$$

$$\mathcal{R} = -2\frac{l}{r} \sqrt{\gamma} \left( \frac{r}{l} \partial_r K - K^2 - (\pi^{r\phi})^2 + \frac{1}{l^2} \right), \quad (2.49)$$

where the extrinsic curvature is given by  $K = -\frac{r}{2l}\gamma^{-1}\partial_r\gamma$  (see appendix A). This gives us a set of four differential equation in  $r$  for which  $P, J, M$  and  $Q$  are the corresponding four integration constants. This can be seen easily as this system is solvable explicitly.

In term of  $L_\pm = K + \frac{1}{l} \pm \pi^{r\phi}$ , we can rewrite the constraints (2.48) and (2.49) as

$$\frac{r}{l}\partial_r L_\pm + \frac{2}{l}L_\pm - L_\pm^2 = 0. \quad (2.50)$$

This gives

$$L_\pm = \frac{2}{l} \frac{A_\pm}{A_\pm + \frac{r^2}{l^2}} = 2l \frac{A_\pm}{r^2} + O(r^{-4}), \quad (2.51)$$

where  $A^\pm$  are two integration constants. We can then solve for  $\pi^{rr}$  and  $\gamma$ :

$$\pi^{rr} = r \frac{P}{2l} + \frac{r}{2l} \left( \frac{\partial_\phi A_+}{A_+ + \frac{r^2}{l^2}} - \frac{\partial_\phi A_-}{A_- + \frac{r^2}{l^2}} \right) = r \frac{P}{2l} + O(r^{-1}), \quad (2.52)$$

$$\gamma = \bar{\gamma} r^2 \left( 1 + \frac{l^2}{r^2} A_+ \right) \left( 1 + \frac{l^2}{r^2} A_- \right) = \bar{\gamma} r^2 + O(1), \quad (2.53)$$

with the last two integration constants  $\bar{\gamma} = \frac{Q^2}{4}$  and  $P$ . The functions  $A_\pm$  are related to  $M$  and  $J$  by:

$$J = 2\bar{\gamma}(A_+ - A_-), \quad M = 2\sqrt{\bar{\gamma}}(A_+ + A_-). \quad (2.54)$$

**Theorem 2.3.** *The four functions  $P, J, M$  and  $Q > 0$  completely determine the configuration asymptotically up to gauge transformations. They parametrize the only degrees of freedom of the theory: the boundary gravitons.*

The above analysis was only done asymptotically. For specific values of  $P, J, M$  and  $Q$ , we have no guaranty that the configuration will be regular everywhere in the bulk.

The BTZ black-holes [2, 3] are given by:

$$P = 0, \quad J = 8G \frac{j}{l}, \quad M = 8G m, \quad Q = 2, \quad (2.55)$$

where  $m$  and  $j$  are the mass and angular momentum of black-hole. Let's remark that we are only talking about a configuration at fixed  $t$ . To have the full 3D black-hole, we also need the right time evolution: the right Hamiltonian. This will be studied in section 4.2.

### 2.3 Dirac bracket for the boundary gravitons

The Poisson bracket of two differentiable functionals  $F[g_{ij}, \pi^{ij}]$  and  $G[g_{ij}, \pi^{ij}]$  is given by

$$\{F[g_{ij}, \pi^{ij}], G[g_{ij}, \pi^{ij}]\} = 16\pi G \int_\Sigma d^2x \left( \frac{\delta F}{\delta g_{ij}} \frac{\delta G}{\delta \pi^{ij}} - \frac{\delta G}{\delta g_{ij}} \frac{\delta F}{\delta \pi^{ij}} \right). \quad (2.56)$$

For differentiable gauge generators, a straightforward computation gives

$$\{\Gamma_\xi, \Gamma_\zeta\} = \tilde{G}[[\xi, \zeta]_g] \quad (2.57)$$

$$\begin{aligned} & + \frac{1}{16\pi G} \oint_{\partial\Sigma} (d^{n-1}x)_k \left\{ 2(\zeta^k \nabla_i \xi_j - \xi^k \nabla_i \zeta_j) \pi^{ij} + 2[\xi, \zeta]_{SD}^j \pi_j^k \right. \\ & \quad + 2\sqrt{g} (\nabla_i \xi^k \nabla^i \zeta - \nabla_i \xi^i \nabla^k \zeta - \nabla_i \zeta^k \nabla^i \xi + \nabla_i \zeta^i \nabla^k \xi) \\ & \quad \left. - (\zeta \zeta^k - \xi \xi^k) (2\Lambda\sqrt{g} - \frac{1}{\sqrt{g}}(\pi^2 - \pi^{ij} \pi_{ij})) + \Theta^k(\mathcal{R}, \mathcal{R}_i) \right\}, \\ [\xi, \zeta]_g^a &= [\xi, \zeta]_{SD}^a + \delta_\zeta \xi^a - \delta_\xi \zeta^a + \Xi^a(\mathcal{R}, \mathcal{R}_i), \end{aligned} \quad (2.58)$$

where  $\xi^a = (\xi, \xi^i)$  and the functions  $\Theta$  and  $\Xi$  are local functions of the constraints and their derivatives. The surface deformation bracket is given by:

$$[\xi, \zeta]_{SD} = \xi^i \partial_i \zeta - \zeta^i \partial_i \xi, \quad (2.59)$$

$$[\xi, \zeta]_{SD}^i = \xi^j \partial_j \zeta^i - \zeta^j \partial_j \xi^i + g^{ij} (\xi \partial_j \zeta - \zeta \partial_j \xi). \quad (2.60)$$

Differentiable gauge generators are first-class functionals, evaluating their Poisson bracket will also give us their Dirac bracket when evaluated on the reduced phase-space. Let's consider two differentiable gauge generators  $\Gamma_1$  and  $\Gamma_2$  associated to the functionals

$$\frac{l}{16\pi G} \oint_{\partial\Sigma} d\phi k_1(P, J, M, Q) \quad \text{and} \quad \frac{l}{16\pi G} \oint_{\partial\Sigma} d\phi k_2(P, J, M, Q). \quad (2.61)$$

The corresponding gauge parameters  $\xi_1$  and  $\xi_2$  are given, up to proper gauge transformations, by the identifications (2.37) and (2.38). By construction, we then have the following

$$\left\{ \frac{l}{16\pi G} \oint_{\partial\Sigma} d\phi k_1(P, J, M, Q), \frac{l}{16\pi G} \oint_{\partial\Sigma} d\phi k_2(P, J, M, Q) \right\}^* \approx \{\Gamma_1, \Gamma_2\}, \quad (2.62)$$

where the LHS is the bracket on the reduced phase space. On the constraints surface, the RHS reduces to the boundary term of (2.57). It is a gauge invariant quantity, it is easier to evaluate it when the gauge is fixed. Using the Fefferman-Graham gauge described in the previous section, we obtain

$$\begin{aligned} \{\Gamma_1, \Gamma_2\} \approx & \frac{l}{16\pi G} \oint_{\partial\Sigma} d\phi \left\{ P(Y_1 \partial_\phi \psi_2 + f_1 \tilde{\chi}_2) + JY_1 \partial_\phi Y_2 + \frac{4J}{Q^2} f_1 \partial_\phi f_2 \right. \\ & + M(Y_1 \partial_\phi f_2 + \psi_1 f_2) + Q(Y_1 \partial_\phi \tilde{\chi}_2 - \psi_1 \tilde{\chi}_2) \\ & \left. + \frac{4}{Q} \partial_\phi \psi_1 \partial_\phi f_2 - (1 \leftrightarrow 2) \right\}. \end{aligned} \quad (2.63)$$

If we replace,  $Y, f, \psi$  and  $\tilde{\chi}$  by their values in term of the Euler-Lagrange derivatives of

$k_1$  and  $k_2$  using (2.37) and (2.38), we obtain the induced Dirac bracket as

$$\begin{aligned} \{\Gamma_1, \Gamma_2\} \approx & \frac{l}{16\pi G} \oint_{\partial\Sigma} d\phi \left\{ P \left( \frac{\bar{\delta}k_1}{\delta J} \partial_\phi \frac{\bar{\delta}k_2}{\delta P} + \frac{\bar{\delta}k_1}{\delta M} \frac{\bar{\delta}k_2}{\delta Q} \right) \right. \\ & + M \left( \frac{\bar{\delta}k_1}{\delta J} \partial_\phi \frac{\bar{\delta}k_2}{\delta M} + \frac{\bar{\delta}k_1}{\delta P} \frac{\bar{\delta}k_2}{\delta M} \right) + Q \left( \frac{\bar{\delta}k_1}{\delta J} \partial_\phi \frac{\bar{\delta}k_2}{\delta Q} - \frac{\bar{\delta}k_1}{\delta P} \frac{\bar{\delta}k_2}{\delta Q} \right) \\ & \left. + J \left( \frac{\bar{\delta}k_1}{\delta J} \partial_\phi \frac{\bar{\delta}k_2}{\delta J} + \frac{4}{Q^2} \frac{\bar{\delta}k_1}{\delta M} \partial_\phi \frac{\bar{\delta}k_2}{\delta M} \right) + \frac{4}{Q} \partial_\phi \frac{\bar{\delta}k_1}{\delta P} \partial_\phi \frac{\bar{\delta}k_2}{\delta M} - (1 \leftrightarrow 2) \right\}. \end{aligned} \quad (2.64)$$

### 3 Boundary hamiltonian

As shown in [14], the differentiable Hamiltonian is given by the boundary conditions on the Lagrange multipliers. More precisely, the Hamiltonian for 3D gravity is given by the differentiable gauge generator associated to the gauge parameters  $N$  and  $N^i$ . We saw in section 2.1 that, on the constraints surface, it is given by a boundary term

$$H[g_{ij}, \pi^{ij}] \approx \frac{l}{16\pi G} \oint_{\partial\Sigma} k_H(M, J, P, Q), \quad (3.1)$$

with

$$f_H \equiv \lim_{r \rightarrow \infty} \frac{N}{r} = \frac{\bar{\delta}k_H}{\delta M}, \quad \psi_H \equiv \lim_{r \rightarrow \infty} \frac{N^r}{r} = \frac{\bar{\delta}k_H}{\delta P}, \quad (3.2)$$

$$Y_H \equiv \lim_{r \rightarrow \infty} N^\phi = \frac{\bar{\delta}k_H}{\delta J}, \quad \tilde{\chi}_H \equiv \lim_{r \rightarrow \infty} \frac{r}{l} \left( \frac{1}{\lambda} \partial_r N - \frac{1}{l} N - \frac{r}{l} \lambda^\phi \partial_\phi N \right) = \frac{\bar{\delta}k_H}{\delta Q}. \quad (3.3)$$

Tuning these boundary conditions we can build any functional  $k_H$  on the boundary. This is our main result:

**Theorem 3.1.** *If we assume that the canonical variables have the following asymptotic behavior:*

$$g_{rr} = \frac{l^2}{r^2} + O(r^{-4}), \quad g_{r\phi} = O(r^{-1}), \quad g_{\phi\phi} = r^2 \bar{\gamma}(t, \phi) + O(1), \quad (3.4)$$

$$\pi^{rr} = O(r), \quad \pi^{r\phi} = O(r^{-2}), \quad \pi^{\phi\phi} = O(r^{-5}), \quad (3.5)$$

$$\mathcal{R} = O(r^{-n}), \quad \mathcal{R}_i = O(r^{-n}) \quad \forall n \in \mathbb{R}. \quad (3.6)$$

*then the set of possible boundary conditions at spatial infinity on the lagrange multipliers  $(N, N^i)$  is in one to one correspondance with the functionals  $\oint_{\partial\Sigma} k_H(M, J, P, Q)$  (modulo the constant functionals) where the boundary fields are defined by:*

$$P(t, \phi) \equiv 2l \lim_{r \rightarrow \infty} \frac{\pi^{rr}}{r}, \quad J(t, \phi) \equiv \frac{2}{l} \lim_{r \rightarrow \infty} \pi_\phi^r, \quad (3.7)$$

$$M(t, \phi) \equiv \frac{2\sqrt{\bar{\gamma}}}{l} \lim_{r \rightarrow \infty} \left( r^2 \left( K + \frac{1}{l} \right) \right), \quad Q(t, \phi) \equiv 2\sqrt{\bar{\gamma}}, \quad (3.8)$$

wich  $\bar{\gamma} > 0$ . On the constraint's surface, we obtain a theory on the boundary  $\partial\Sigma$  with a phase-space parametrized by  $(M, J, P, Q)$  with a bracket given in equation (2.64) and an Hamiltonian given by

$$H[g_{ij}, \pi^{ij}] \approx \frac{l}{16\pi G} \oint_{\partial\Sigma} k_H(M, J, P, Q), \quad (3.9)$$

This analysis only concerns the differentiable structure at infinity, we didn't treat any of the possible obstruction coming from the bulk structure of the space-time.

A surprising feature is the need for 4 functions in order to completely describe the asymptotic phase-space. When written in term of Chern-Simons theory, one needs 6 functions to describe the corresponding asymptotic phase-space. Since one adds 3 gauge degrees of freedom in the bulk, one would have expected to have three more asymptotic functions in the Chern-Simons description compared to the metric description.

We will now study the different type of boundary conditions that appeared in the literature. We will start with the sets of boundary conditions that have the conformal algebra in two dimensions as a symmetry algebra.

## 4 Some examples of boundary conditions

### 4.1 Conformal

Let's consider the boundary conditions presented in [7]. With the coordinates  $x^A = t, \phi$ , they are given by:

$$g_{rr} = \frac{l^2}{r^2} + C_{rr}r^{-4} + o(r^{-4}), \quad (4.1)$$

$$g_{rA} = C_{rA}r^{-3} + o(r^{-3}), \quad (4.2)$$

$$g_{AB} = r^2 e^{2\varphi} \eta_{AB} + C_{AB} + o(1), \quad (4.3)$$

$$0 = e^{-2\varphi} \eta^{AB} C_{AB} + \frac{1}{l^2} C_{rr}, \quad (4.4)$$

where  $\eta_{AB} dx^A dx^B = -\frac{1}{l^2} dt^2 + d\phi^2$  is a fixed metric on the cylinder and  $\eta^{AB}$  is its inverse. In term of those fields, our quantities describing the boundary gravitons are given by:

$$Q = 2e^\varphi, \quad M = \frac{2}{l^2} e^\varphi \left( e^{-2\varphi} C_{\phi\phi} + \frac{1}{2l^2} C_{rr} \right), \quad (4.5)$$

$$P = -2l\dot{\varphi}, \quad J = \frac{2}{l} C_{t\phi}. \quad (4.6)$$

The lagrange multipliers take the following form:

$$N = \frac{r}{l} e^\varphi - \frac{l}{2} C_{tt} e^{-\varphi} r^{-1} + o(r^{-1}), \quad N^r = O(r^{-1}), \quad N^\phi = O(r^{-2}), \quad (4.7)$$

which leads to

$$f_H = \frac{Q}{2l}, \quad \tilde{\chi}_H = \frac{M}{2l}, \quad \psi_H = 0, \quad Y_H = 0. \quad (4.8)$$

The associated differentiable Hamiltonian is then easily computed

$$H_{EBH} \approx \frac{1}{16\pi G} \oint_{\partial\Sigma} d\phi \frac{1}{2} M Q. \quad (4.9)$$

For the BTZ Black-hole (2.55), we have  $H_{EBH} \approx m$  as expected. Using the equation of motion for  $Q$ , one can check that the condition  $Q > 0$  is preserved under time evolution.

This set of boundary conditions possesses an asymptotic symmetry group given by two Virasoros in semi-direct product with two current algebras. As we have already computed the induced bracket on the boundary gravitons, we just need to find the boundary generators in terms of  $Q, P, J$  and  $M$  that are symmetry generators for the Hamiltonian  $H_{EBH}$ . Let's define

$$\mathcal{L}^\pm(\phi) = \frac{l}{32\pi G} \left( \frac{1}{2} M Q \pm J \right), \quad \mathcal{P}^\pm(\phi) = \frac{l}{32\pi G} \left( -P \pm \frac{2}{Q} \partial_\phi Q \right), \quad (4.10)$$

$$\mathcal{Q} = \frac{-l}{16\pi G} \oint_{\partial\Sigma} d\phi \log Q. \quad (4.11)$$

They have the following bracket

$$\{\mathcal{L}^\pm(\phi), \mathcal{L}^\pm(\phi')\}^* \approx \pm \mathcal{L}^\pm(\phi) \partial_\phi \delta(\phi - \phi') \mp \mathcal{L}^\pm(\phi') \partial_{\phi'} \delta(\phi' - \phi), \quad (4.12)$$

$$\{\mathcal{L}^\pm(\phi), \mathcal{P}^\pm(\phi')\}^* \approx \pm \mathcal{P}^\pm(\phi) \partial_\phi \delta(\phi - \phi') - \frac{l}{16\pi G} \partial_\phi^2 \delta(\phi - \phi'), \quad (4.13)$$

$$\{\mathcal{P}^\pm(\phi), \mathcal{P}^\pm(\phi')\}^* \approx \mp \frac{l}{16\pi G} \partial_\phi \delta(\phi - \phi'), \quad (4.14)$$

$$\{\mathcal{L}^\pm(\phi), \mathcal{Q}\}^* \approx \frac{1}{2} \mathcal{P}^\pm(\phi), \quad \{\mathcal{P}^\pm(\phi), \mathcal{Q}\}^* \approx -\frac{l}{32\pi G}, \quad (4.15)$$

where the rest gives zero. If we expand them in modes,

$$\mathcal{L}_m^\pm = \oint_{\partial\Sigma} d\phi e^{\pm im\phi} \mathcal{L}^\pm(\phi), \quad \mathcal{P}_m^\pm = \oint_{\partial\Sigma} d\phi e^{\pm im\phi} \mathcal{P}^\pm(\phi), \quad (4.16)$$

we recover the algebra obtained in [7]:

$$\begin{aligned} i \{\mathcal{L}_m^\pm, \mathcal{L}_n^\pm\}^* &= (m-n) \mathcal{L}_{m+n}^\pm, & i \{\mathcal{L}_m^+, \mathcal{L}_n^-\}^* &= 0, \\ i \{\mathcal{L}_m^\pm, \mathcal{P}_n^\pm\}^* &= -n \mathcal{P}_{m+n}^\pm + \frac{l}{8G} i m^2 \delta_{m+n,0}, & i \{\mathcal{L}_m^\pm, \mathcal{P}_n^\mp\}^* &= 0, \\ i \{\mathcal{P}_m^\pm, \mathcal{P}_n^\pm\}^* &= -\frac{l}{8G} m \delta_{m+n,0}, & i \{\mathcal{P}_m^+, \mathcal{P}_n^-\}^* &= 0, \\ i \{\mathcal{L}_m^\pm, \mathcal{Q}\}^* &= \frac{i}{2} \mathcal{P}_m^\pm, & i \{\mathcal{P}_m^\pm, \mathcal{Q}\}^* &= -i \frac{l}{16G} \delta_{m,0}. \end{aligned} \quad (4.17)$$

The identification  $\mathcal{P}_0^+ = \mathcal{P}_0^-$  is also present here:

$$\oint_{\partial\Sigma} d\phi \mathcal{P}^+ = \oint_{\partial\Sigma} d\phi \mathcal{P}^- = \frac{-l}{32\pi G} \oint_{\partial\Sigma} d\phi P. \quad (4.18)$$

From this algebra, we can easily reconstruct the conserved quantities  $\mathcal{L}_m^\pm(t)$ ,  $\mathcal{P}_m^\pm(t)$  and  $\mathcal{Q}(t)$  where the quantities defined in (4.10)-(4.11) are their values at  $t = 0$ . A conserved quantity  $F(t)$  satisfies

$$\frac{\partial}{\partial t}F + \{F, H\}^* \approx 0, \quad (4.19)$$

where  $\frac{\partial}{\partial t}$  only hits the explicit dependence on time. Using  $H_{EBH} = \frac{1}{l}(\mathcal{L}_0^+ + \mathcal{L}_0^-)$ , we obtain:

$$\mathcal{L}_m^\pm(t) = e^{im\frac{t}{l}}\mathcal{L}_m^\pm, \quad \mathcal{P}_m^\pm(t) = e^{im\frac{t}{l}}\mathcal{P}_m^\pm, \quad \mathcal{Q}(t) = \mathcal{Q} + \frac{2}{l}\mathcal{P}_0 t. \quad (4.20)$$

By construction, the algebra (4.17) is time independent.

These conserved quantities are associated to asymptotic symmetries using the dictionary given in (2.37)-(2.38). For instance, the angular momentum is

$$\mathcal{L}_0^+ - \mathcal{L}_0^- = \frac{l}{16\pi G} \oint_{\partial\Sigma} d\phi J. \quad (4.21)$$

It leads to

$$f = 0, \quad Y = 1, \quad \psi = 0, \quad \chi = 0, \quad (4.22)$$

and then

$$\xi = O(r^{-3}), \quad \xi^r = O(r^{-1}), \quad \xi^\phi = 1 + O(r^{-2}), \quad (4.23)$$

which is the expected rotation in  $\frac{\partial}{\partial\phi}$  at infinity.

## 4.2 Brown-Henneaux

The original Brown-Henneaux (BH) boundary conditions are a sub-set of the boundary conditions presented in the previous section where part of the boundary degrees of freedom are frozen. We saw in [14] that such additional boundary conditions on the phase-space can be imposed through residual constraints on the boundary.

The BH boundary conditions are given by

$$g_{rr} = \frac{l^2}{r^2} + C_{rr}r^{-4} + o(r^{-4}), \quad (4.24)$$

$$g_{rA} = C_{rA}r^{-3} + o(r^{-3}), \quad (4.25)$$

$$g_{AB} = r^2\eta_{AB} + C_{AB} + o(1). \quad (4.26)$$

Our boundary variables are then easily computed. We have

$$Q = 2, \quad M = \frac{2}{l^2} \left( C_{\phi\phi} + \frac{1}{2l^2} C_{rr} \right), \quad (4.27)$$

$$P = 0, \quad J = \frac{2}{l} C_{t\phi}, \quad (4.28)$$

and, for the lagrange multipliers,

$$N = \frac{r}{l} - \frac{l}{2}C_{tt}r^{-1} + o(r^{-1}), \quad N^r = O(r^{-1}), \quad N^\phi = O(r^{-2}). \quad (4.29)$$

We see that the phase-space is smaller in this case: we have to impose both  $Q = 2$  and  $P = 0$ . The boundary gravitons are then completely parametrized by the boundary fields  $M$  and  $J$ .

In order to describe this phase-space, we will treat the two additional boundary conditions on the boundary variables as constraints. This can be done by relaxing the boundary conditions on the corresponding lagrange multipliers: we have to relax both  $\chi_H$  and  $\psi_H$ . Looking at the asymptotic form of  $N$ , we see that  $\tilde{\chi}_H$  is already relaxed: we have

$$\tilde{\chi}_H = \frac{1}{l}C_{tt} - \frac{1}{2l^5}C_{rr}, \quad (4.30)$$

but, this time,  $C_{tt}$  is not related to  $M$ . Let's consider the following relaxed asymptotics for the lagrange multipliers:

$$N = \frac{r}{l} - \left(\frac{l^2}{2}\tilde{\chi}_H + \frac{1}{4l^3}C_{rr}\right)r^{-1} + o(r^{-1}), \quad (4.31)$$

$$N^r = \psi_H + O(r^{-1}), \quad N^\phi = O(r^{-2}), \quad (4.32)$$

where both  $\tilde{\chi}_H$  and  $\psi_H$  are free to vary. The corresponding differentiable Hamiltonian generating the additional boundary constraints  $P = 0$  and  $Q = 2$  is then given by:

$$H_{BH} = \frac{1}{16\pi G} \int_{\Sigma} d^2x \left( N\mathcal{R} + N^i\mathcal{R}_i \right) + \frac{1}{16\pi G} \oint_{\partial\Sigma} d\phi \left( M + l\psi_H P + l\tilde{\chi}_H(Q - 2) \right), \quad (4.33)$$

The variation of the action then gives

$$\delta S = \int dt \int_{\Sigma} d^2x \left( \frac{\delta S}{\delta g_{ij}} \delta g_{ij} + \frac{\delta S}{\delta \pi^{ij}} \delta \pi^{ij} - \delta N\mathcal{R} - \delta N^i\mathcal{R}_i \right) + \frac{l}{16\pi G} \oint_{\partial\Sigma} d\phi \left( \delta\psi_H P + \delta\tilde{\chi}_H(Q - 2) \right), \quad (4.34)$$

which is what we wanted:  $\psi_H$  and  $\tilde{\chi}_H$  are playing the role of lagrange multipliers enforcing  $Q = 2$  and  $P = 0$ .

We can now do our analysis of the boundary dynamics using the full boundary phase-space described in section 2.3 with the Hamiltonian:

$$H_{BH} \approx \frac{1}{16\pi G} \oint_{\partial\Sigma} d\phi M + \frac{l}{16\pi G} \oint_{\partial\Sigma} d\phi \left( \psi_H P + \tilde{\chi}_H(Q - 2) \right), \quad (4.35)$$

$$\approx H_{EBH} + \frac{l}{16\pi G} \oint_{\partial\Sigma} d\phi \left( \psi_H P + \tilde{\chi}_H(Q - 2) \right). \quad (4.36)$$



In the second line, we used the constraints  $Q = 2$  to recover the Hamiltonian of the previous section (4.9). We see that the theory corresponding to the Brown-Henneaux boundary conditions is a constrained version of the theory associated to the boundary conditions (4.1)-(4.4).

The boundary constraints are second-class:

$$\{P(\phi), Q(\phi') - 2\}^* \approx -\frac{16\pi G}{l} \left( Q(\phi) - 2 \right) \delta(\phi - \phi') - \frac{32\pi G}{l} \delta(\phi - \phi'), \quad (4.37)$$

the other brackets being zero. It is then straightforward to compute the induced bracket on the fully reduced phase-space. In term of  $M$  and  $J$ , we have

$$\{M(\phi), M(\phi')\}^{**} \approx \frac{16\pi G}{l} \left( J(\phi) \partial_\phi \delta(\phi - \phi') - J(\phi') \partial_{\phi'} \delta(\phi' - \phi) \right), \quad (4.38)$$

$$\begin{aligned} \{M(\phi), J(\phi')\}^{**} \approx & \frac{16\pi G}{l} \left( M(\phi) \partial_\phi \delta(\phi - \phi') - M(\phi') \partial_{\phi'} \delta(\phi' - \phi) \right) \\ & - \frac{32\pi G}{l} \partial_\phi^3 \delta(\phi - \phi'), \end{aligned} \quad (4.39)$$

$$\{J(\phi), J(\phi')\}^{**} \approx \frac{l}{16\pi G} \left( J(\phi) \partial_\phi \delta(\phi - \phi') - J(\phi') \partial_{\phi'} \delta(\phi' - \phi) \right), \quad (4.40)$$

where  $\approx$  means in this case that we have imposed all constraints: from both the bulk and the boundary. On this fully reduced phase-space, the Hamiltonian is simply given by

$$H_{BH} \approx \frac{1}{16\pi G} \oint_{\partial\Sigma} d\phi M. \quad (4.41)$$

The two Virasoro algebras of conserved charges can be recovered easily. Defining

$$\overline{\mathcal{L}}^\pm(\phi) = \frac{l}{32\pi G} \left( M(\phi) \pm J(\phi) \right), \quad \overline{\mathcal{L}}_m^\pm = \oint_{\partial\Sigma} d\phi e^{\pm im\phi} \overline{\mathcal{L}}^\pm(\phi), \quad (4.42)$$

we obtain the usual result

$$i \left\{ \overline{\mathcal{L}}_m^\pm, \overline{\mathcal{L}}_n^\pm \right\}^{**} \approx (m - n) \overline{\mathcal{L}}_{m+n}^\pm + \frac{l}{8G} m^3 \delta_{m+n,0}, \quad i \left\{ \overline{\mathcal{L}}_m^+, \overline{\mathcal{L}}_n^- \right\}^{**} \approx 0. \quad (4.43)$$

The Virasoro generators  $\overline{\mathcal{L}}_m^\pm$  are the generators defined on the previous section  $\mathcal{L}_m^\pm$  evaluated on the constraint's surface  $Q = 2$  and  $P = 0$ . The central charge in (4.43) appeared due to the correction coming from the Dirac bracket  $\{, \}^{**}$ . The conserved charges  $\mathcal{L}_m^\pm(t)$  are easily computed:

$$\overline{\mathcal{L}}_m^\pm(t) = e^{im\frac{t}{l}} \overline{\mathcal{L}}_m^\pm. \quad (4.44)$$

The algebra obtained here is of course just the current algebra of the dual Liouville theory living on the boundary [10, 11, 12, 13].

### 4.3 Chiral

In [8], the authors proposed a set of chiral boundary conditions for  $AdS_3$  that was extended in [9]. We will first find the Hamiltonian for the extended version and then obtain the additional boundary constraints corresponding to the original chiral boundary conditions. For the extended case, the asymptotic behavior of the metric in the Fefferman-Graham gauge can be written as

$$g_{rr} = \frac{l^2}{r^2}, \quad (4.45)$$

$$g_{r\phi} = 0, \quad (4.46)$$

$$g_{\phi\phi} = r^2(1 + F) + C_{\phi\phi} + o(1), \quad (4.47)$$

$$g_{rt} = 0, \quad (4.48)$$

$$g_{t\phi} = \frac{F}{l}r^2 + C_{t\phi} + o(1), \quad (4.49)$$

$$g_{tt} = \frac{r^2}{l^2}(-1 + F) + \Delta - l^{-2}C_{\phi\phi} + 2l^{-1}C_{t\phi} + o(1), \quad (4.50)$$

where  $F$  is a function of  $t$  and  $\phi$  and  $\Delta$  is a fixed constant. As we assumed  $\bar{\gamma} > 0$ , this means that we are studying the case  $F > -1$  only. A straightforward computation leads to the following values for our quantities describing the boundary gravitons:

$$Q = 2\sqrt{1 + F}, \quad M = \frac{2}{l^2} \frac{C_{\phi\phi}}{\sqrt{1 + F}} + \frac{1}{l^4} C_{rr} \sqrt{1 + F}, \quad (4.51)$$

$$P = -l\partial_t F + \frac{2 + F}{1 + F} \partial_\phi F, \quad J = \frac{2}{l} C_{t\phi}(1 + F) - \frac{2}{l^2} C_{\phi\phi} F, \quad (4.52)$$

associated to the lagrange multipliers:

$$N = \frac{r}{l} \frac{1}{\sqrt{1 + F}} + \frac{l}{2r} \sqrt{1 + F} \left( -\Delta + \frac{C_{\phi\phi}}{l^2} \frac{1 + 2F}{(1 + F)^2} - \frac{2}{l} \frac{C_{t\phi}}{1 + F} \right) + o(r^{-1}), \quad (4.53)$$

$$N^\phi = \frac{1}{l} \frac{F}{(1 + F)} + O(r^{-2}), \quad N^r = O(r^{-1}). \quad (4.54)$$

This leads to

$$f_H = \frac{2}{l} \frac{1}{Q}, \quad Y_H = \frac{1}{l} - \frac{4}{l} \frac{1}{Q^2}, \quad \Psi_H = 0, \quad (4.55)$$

$$\chi_H = \frac{\Delta}{2l} Q + \frac{8}{l} \frac{J}{Q^3} - \frac{2}{l} \frac{M}{Q^2}, \quad (4.56)$$

and to the Hamiltonian:

$$H_{EC} \approx \frac{1}{16\pi G} \oint_{\partial\Sigma} d\phi \left( J - 4 \frac{J}{Q^2} + 2 \frac{M}{Q} + \Delta \frac{Q^2}{4} \right). \quad (4.57)$$

The equation of motion for  $F$  is given by

$$\Delta \frac{\partial}{\partial x^-} F + \left( \frac{\partial}{\partial x^-} \right)^3 F = 0. \quad (4.58)$$

where  $x^- = \frac{t}{l} - \phi$ . In general, we cannot expect the time evolution to preserve the condition  $F > -1$ . The breaking of this condition means that the surfaces of constant  $t$  are not space-like and our ADM split is not valid anymore. However, if the initial conditions satisfy  $F > -1$  then it will stay valid close to  $t_0$  and, in this neighborhood, we can still apply our analysis.

In [9], the authors showed that, for  $\Delta < 0$ , the algebra of the charges is given by the semi-direct product of a Virasoro algebra with a  $sl(2, \mathbb{R})$  current algebra. Functionals of the boundary gravitons reproducing this result are built from

$$L_C(\phi) = \frac{l}{16\pi G} J - \partial_\phi \mathcal{P}^+, \quad T_C^0(\phi) = \mathcal{P}^+, \quad (4.59)$$

$$T_C^+(\phi) = \frac{l}{4\pi G} \left( \frac{M}{Q} - 2 \frac{J}{Q^2} \right), \quad T_C^-(\phi) = \frac{l}{16\pi G} \frac{-Q^2}{8}, \quad (4.60)$$

where  $\mathcal{P}^+(\phi)$  was defined in (4.10). The brackets of these new quantities are given by:

$$\begin{aligned} \{L_C(\phi), L_C(\phi')\}^* &\approx L_C(\phi) \partial_\phi \delta(\phi - \phi') - L_C(\phi') \partial_{\phi'} \delta(\phi' - \phi) \\ &\quad - \frac{l}{16\pi G} \partial_\phi^3 \delta(\phi - \phi'), \end{aligned} \quad (4.61)$$

$$\{L_C(\phi), T_C^a(\phi')\}^* \approx T_C^a(\phi) \partial_\phi \delta(\phi - \phi'), \quad (4.62)$$

$$\{T_C^a(\phi), T_C^b(\phi')\}^* \approx f_c^{ab} T_C^c(\phi) \delta(\phi - \phi') + \frac{l}{16\pi G} \eta^{ab} \partial_\phi \delta(\phi - \phi'), \quad (4.63)$$

where  $a, b, c = +, -, 0$ . The current algebra is characterized by

$$f_+^{0+} = -1, \quad f_-^{0-} = 1, \quad f_0^{+-} = 2, \quad \eta^{00} = -1, \quad \eta^{+-} = 2, \quad (4.64)$$

with all the other components equal to zero. If we develop in modes:

$$L_m = \oint_{\partial\Sigma} d\phi e^{im\phi} L_C(\phi), \quad T_m^a = \oint_{\partial\Sigma} d\phi e^{im\phi} T_C^a(\phi), \quad (4.65)$$

we recover the algebra of the charges found in [9]

$$i \{L_m, L_n\} = (m - n) L_{m+n} + \frac{l}{8G} m^3 \delta_{m+n,0}, \quad (4.66)$$

$$i \{L_m, T_n^a\} = -n T_{m+n}^a, \quad (4.67)$$

$$i \{T_m^a, T_n^b\} = i f_c^{ab} T_{m+n}^c + \frac{l}{8G} \eta^{ab} m \delta_{m+n,0}. \quad (4.68)$$

In terms of these generators, the Hamiltonian is given by,

$$H = \frac{1}{l} (L_0 + \frac{1}{2} T_0^+ - 2\Delta T_0^-). \quad (4.69)$$

The conserved quantities are easily built by adding an explicit time dependence to each mode. If  $\Delta = -\alpha^2$ , we get

$$\tilde{L}_m(t) = L_m e^{im\frac{t}{l}}, \quad (4.70)$$

$$\begin{aligned} \tilde{T}_m^0(t) &= T_m^0 e^{im\frac{t}{l}} \cos(2\alpha\frac{t}{l}) + \frac{1}{4\alpha} T_m^+ e^{im\frac{t}{l}} \sin(2\alpha\frac{t}{l}) \\ &\quad - \alpha T_m^- e^{im\frac{t}{l}} \sin(2\alpha\frac{t}{l}), \end{aligned} \quad (4.71)$$

$$\begin{aligned} \tilde{T}_m^+(t) &= T_m^+ e^{im\frac{t}{l}} \cos^2(\alpha\frac{t}{l}) + 4\alpha^2 T_m^- e^{im\frac{t}{l}} \sin^2(\alpha\frac{t}{l}) \\ &\quad - 2\alpha T_m^0 e^{im\frac{t}{l}} \sin(2\alpha\frac{t}{l}), \end{aligned} \quad (4.72)$$

$$\begin{aligned} \tilde{T}_m^-(t) &= \frac{1}{4\alpha^2} T_m^+ e^{im\frac{t}{l}} \sin^2(\alpha\frac{t}{l}) + T_m^- e^{im\frac{t}{l}} \cos^2(\alpha\frac{t}{l}) \\ &\quad + \frac{1}{2\alpha} T_m^0 e^{im\frac{t}{l}} \sin(2\alpha\frac{t}{l}). \end{aligned} \quad (4.73)$$

The cases  $\Delta = 0$  and  $\Delta < 0$  can be obtained in a similar way.

The charges we obtained here are not the one obtained in [9]. However, we built these because they are well adapted to the constraints analysis that we will do in the next section.

## 4.4 Constrained Chiral

The original chiral boundary conditions introduced in [8] are a subset of the one introduced in the previous section with the additional condition

$$\partial_t F - \frac{1}{l} \partial_\phi F = 0. \quad (4.74)$$

A good point here is that this extra condition guarantees the preservation of  $F > -1$  under time evolution. This can easily be rewritten as

$$T_C^0 = \mathcal{P}^+ = \frac{l}{32\pi G} \left( -P + \frac{2}{Q} \partial_\phi Q \right) = 0. \quad (4.75)$$

The boundary theory associated to these restricted boundary conditions can be described by a theory built from the Hamiltonian (4.57) with the constraint  $T_C^0 \approx 0$ .

For simplicity, the rest of the analysis will be done using Fourier modes. The primary constraints are  $T_m^0 \approx 0$  for all  $m$ . They lead to secondary constraints:

$$\{T_m^0, lH\} = -imT_m^0 - \frac{1}{2}T_m^+ - 2\Delta T_m^-, \quad (4.76)$$

$$\Rightarrow K_m^+ \equiv \frac{1}{2}T_m^+ + 2\Delta T_m^- \approx 0. \quad (4.77)$$

With these extra constraints, the set is complete:

$$\{K_m^+, lH\} = -imK_m^+ - 4\Delta T_m^0 \approx 0. \quad (4.78)$$

Their algebra is complicated and it is difficult to form the Dirac bracket. However, when  $\Delta \neq 0$ ,  $L_m$  and  $K_m^- \equiv \frac{1}{2}T_m^+ - 2\Delta T_m^-$  form a complete set of gauge-invariant quantities. The reduced phase-space is parametrized by  $L_m$  and  $K_m^-$  with

$$i \{L_m, L_n\} = (m-n)L_{m+n} + \frac{l}{8G}m^3\delta_{m+n,0}, \quad (4.79)$$

$$i \{L_m, K_n^-\} = -nK_{m+n}^-, \quad (4.80)$$

$$i \{K_m^-, K_n^-\} = -\frac{\Delta l}{2G}m\delta_{m+n,0}, \quad (4.81)$$

and the Hamiltonian given by

$$H = \frac{1}{l}(L_0 + K_0^-). \quad (4.82)$$

To make the link with the charges obtained in equations (2.15) and (2.16) of [8], we define

$$\hat{L}_m = L_m + \frac{1}{2}K_m^- - \frac{l\Delta}{16G}\delta_{m,0}, \quad \hat{P}_m = \frac{1}{2}K_m^- - \frac{l\Delta}{16G}\delta_{m,0}. \quad (4.83)$$

which leads to the following algebra

$$i \{\hat{L}_m, \hat{L}_n\} = (m-n)\hat{L}_{m+n} + \frac{l}{8G}m^3\delta_{m+n,0}, \quad (4.84)$$

$$i \{\hat{L}_m, \hat{P}_n\} = -n\hat{P}_{m+n} - \frac{\Delta l}{16G}m\delta_{m+n,0}, \quad (4.85)$$

$$i \{\hat{P}_m, \hat{P}_n\} = -\frac{\Delta l}{8G}m\delta_{m+n,0}. \quad (4.86)$$

The algebra we obtain here is the algebra of the generators written in equations (2.15) and (2.16) of [8]. However, the difference between the extension obtained here and the one written in equations (2.17)-(2.19) of [8] is a redefinition of the zero mode  $\hat{P}_0 \rightarrow \hat{P}_0 - \frac{l\Delta}{16G}\delta_{m,0}$ . This change of basis absorbs the extension in (4.85) and brings back the algebra to the canonical form with the following central charge and level

$$c_R = \frac{3l}{2G}, \quad k_{KM} = -\frac{l\Delta}{4G}. \quad (4.87)$$

## Acknowledgements

I would like to thank G. Barnich, H. González, P. Ritter and D. Tempo for useful discussions. This work is founded by the fundecyt postdoctoral grant 3140125. The Centro de Estudios Científicos (CECs) is funded by the Chilean Government through the Centers of Excellence Base Financing Program of Conicyt.

## A Radial decomposition

Let's assume that we have spatial coordinates given by  $x^i = (r, \phi)$ . We introduce:

$$\gamma \equiv g_{\phi\phi}, \quad \lambda_\phi \equiv g_{r\phi}, \quad \lambda^\phi \equiv \lambda_\phi \gamma^{-1}, \quad \lambda \equiv \frac{1}{\sqrt{g^{rr}}}. \quad (\text{A.1})$$

The metric and its inverse take the form:

$$g_{ij} = \begin{pmatrix} \lambda^2 + \lambda^\phi \lambda_\phi & \lambda_\phi \\ \lambda_\phi & \gamma \end{pmatrix}, \quad g^{ij} = \begin{pmatrix} \frac{1}{\lambda^2} & -\frac{\lambda^\phi}{\lambda^2} \\ -\frac{\lambda^\phi}{\lambda^2} & \gamma^{-1} + \frac{\lambda^\phi \lambda_\phi}{\lambda^2} \end{pmatrix}, \quad (\text{A.2})$$

where we used  $\gamma$  and its inverse  $\gamma^{-1}$  to raise and lower the angular indices  $\phi$

Introducing the extrinsic curvature of the (1)-spheres  $K_{\phi\phi}$ , we can write all the Christoffel symbols:

$$K_{\phi\phi} = \frac{1}{2\lambda} (-\partial_r \gamma + 2D_\phi \lambda_\phi) \quad (\text{A.3})$$

$$K = \gamma^{-1} K_{\phi\phi} = \frac{1}{2\lambda} ((-\partial_r + \lambda^\phi \partial_\phi) \log \gamma + 2\partial_\phi \lambda^\phi), \quad (\text{A.4})$$

$$\Gamma_{\phi\phi}^r = \frac{1}{\lambda} \gamma K \quad (\text{A.5})$$

$$\Gamma_{\phi\phi}^\phi = \frac{1}{2} \partial_\phi \log \gamma - \frac{\lambda_\phi}{\lambda} K \quad (\text{A.6})$$

$$\Gamma_{r\phi}^r = \frac{1}{\lambda} (\partial_\phi \lambda + K \lambda_\phi) \quad (\text{A.7})$$

$$\Gamma_{rr}^r = \frac{1}{\lambda} \partial_r \lambda + \frac{\lambda^\phi}{\lambda} (\partial_\phi \lambda + K \lambda_\phi) \quad (\text{A.8})$$

$$\Gamma_{r\phi}^\phi = -\frac{\lambda^\phi}{\lambda} (\partial_\phi \lambda + K \lambda_\phi) + D_\phi \lambda^\phi - \lambda K \quad (\text{A.9})$$

$$\Gamma_{rr}^\phi = -\lambda \left( \gamma^{-1} + \frac{\lambda^\phi \lambda_\phi}{\lambda^2} \right) (\partial_\phi \lambda + K \lambda_\phi) - \lambda^\phi (D_\phi \lambda^\phi - \lambda K) - \frac{\lambda^\phi}{\lambda} \partial_r \lambda + \gamma^{-1} \partial_r \lambda_\phi \quad (\text{A.10})$$

where  $D_\phi$  is the covariant derivative associated to  $\gamma$ .

We have introduced new variables to describe  $g_{ij}$ . Some computations can be simplified introducing their canonical conjugates:

$$\theta \equiv 2\lambda \pi^{rr}, \quad \theta_\phi \equiv 2\gamma \pi^{r\phi} + 2\lambda_\phi \pi^{rr}, \quad \sigma \equiv \pi^{\phi\phi} + 2\lambda^\phi \pi^{\phi r} + \lambda^\phi \lambda^\phi \pi^{rr}. \quad (\text{A.11})$$

They satisfy:

$$\pi^{ij} \delta g_{ij} = \theta \delta \lambda + \theta_\phi \delta \lambda^\phi + \sigma \delta \gamma, \quad (\text{A.12})$$

$$\{\lambda(x), \theta(x')\} = \delta^2(x - x'), \quad \{\lambda^\phi(x), \theta_\phi(x')\} = \delta^2(x - x'), \quad (\text{A.13})$$

$$\{\gamma(x), \sigma(x')\} = \delta^2(x - x'), \quad (\text{A.14})$$

all the other Poisson brackets being zero.

## B Gauge transformations

It is useful to have the gauge transformation laws of the new dynamical fields defined in appendix A. We will only consider here transformations with parameters  $\xi, \xi^i$  independent of the canonical variables. Straightforward but long computations lead to

- for the metric, we obtain:

$$\delta_\xi \lambda = -\xi \sqrt{\gamma} \sigma + \partial_r(\lambda \xi^r) + \xi^\phi \partial_\phi \lambda - \lambda \lambda^\phi \partial_\phi \xi^r, \quad (\text{B.1})$$

$$\begin{aligned} \delta \lambda^\phi &= \lambda \xi \gamma^{-\frac{3}{2}} \theta_\phi + \partial_r(\xi^\phi + \xi^r \lambda^\phi) \\ &\quad - \lambda^\phi \partial_\phi \xi^\phi + \xi^\phi \partial_\phi \lambda^\phi + \partial_\phi \xi^r (\lambda^2 \gamma^{-1} - (\lambda^\phi)^2), \end{aligned} \quad (\text{B.2})$$

$$\delta_\xi \gamma = -\xi \sqrt{\gamma} \theta + \xi^i \partial_i \gamma + 2 \partial_\phi \xi^\phi \gamma + 2 \partial_\phi \xi^r \lambda_\phi, \quad (\text{B.3})$$

- for the trace of the extrinsic curvature  $K$

$$\begin{aligned} \delta_\xi K &= \frac{\xi}{\lambda} \sqrt{\gamma} \sigma K - \frac{1}{2\lambda} (\lambda^\phi \partial_\phi - \partial_r) \left( \frac{\theta \xi}{\sqrt{\gamma}} \right) + \frac{1}{\lambda} D_\phi \left( \lambda \xi \frac{\theta^\phi}{\sqrt{\gamma}} \right) \\ &\quad + \xi^r \partial_r K + 2 \partial_\phi \lambda \gamma^{-1} \partial_\phi \xi^r + \lambda D_\phi D^\phi \xi^r + \xi^\phi \partial_\phi K. \end{aligned} \quad (\text{B.4})$$

- the variations of the momenta are given by

$$\begin{aligned} \delta_\xi \theta &= -2\sqrt{\gamma} \xi \Lambda - 2\sqrt{\gamma} D_\phi D^\phi \xi + \frac{2}{\lambda} \sqrt{\gamma} K (\partial_r - \lambda^\phi \partial_\phi) \xi - \frac{\xi}{2} \sqrt{\gamma} (\gamma^{-1} \theta_\phi)^2 \\ &\quad + \xi^r \partial_r \theta + \partial_\phi (\xi^\phi \theta) + \lambda^\phi \partial_\phi \xi^r \theta - 2\lambda \partial_\phi \xi^r \gamma^{-1} \theta_\phi, \end{aligned} \quad (\text{B.5})$$

$$\begin{aligned} \delta_\xi \theta_\phi &= 2\sqrt{\gamma} \partial_\phi \left[ \frac{1}{\lambda} (\partial_r - \lambda^\phi \partial_\phi) \xi \right] + 2\sqrt{\gamma} K \partial_\phi \xi \\ &\quad + \xi^r \partial_r \theta_\phi + \partial_\phi (\xi^\phi \theta_\phi) + \partial_\phi \xi^\phi \theta_\phi + 2 \partial_\phi \xi^r \left( \frac{\lambda \theta}{2} - \gamma \sigma + \lambda^\phi \theta_\phi \right), \end{aligned} \quad (\text{B.6})$$

$$\begin{aligned} \delta_\xi \sigma &= \partial_i (\xi^i \sigma) - 2\sigma (\partial_\phi \xi^\phi + \lambda^\phi \partial_\phi \xi^r) + \lambda^2 \gamma^{-2} \partial_\phi \xi^r \theta_\phi - \frac{\xi}{\sqrt{\gamma}} \lambda \Lambda \\ &\quad + \frac{1}{2} \frac{\xi}{\sqrt{\gamma}} \sigma \theta + \frac{3}{4} \frac{\xi}{\sqrt{\gamma}} \lambda (\gamma^{-1} \theta_\phi)^2 - \frac{1}{\sqrt{\gamma}} \partial_\phi \lambda \gamma^{-1} \partial_\phi \xi \\ &\quad - \frac{1}{\sqrt{\gamma}} (\partial_r - \lambda^\phi \partial_\phi) \left[ \frac{1}{\lambda} (\partial_r - \lambda^\phi \partial_\phi) \xi \right], \end{aligned} \quad (\text{B.7})$$

In section 2.1, we showed that the gauge transformations preserving the asymptotics are given by

$$\xi = r f + \kappa + O(r^{-3}), \quad (\text{B.8})$$

$$\xi^r = r \psi + O(r^{-1}), \quad (\text{B.9})$$

$$\xi^\phi = Y + O(r^{-3}), \quad (\text{B.10})$$

where

$$\kappa = -\frac{l^2}{2r}\chi - r \int_r^\infty dr' j(r') = O(r^{-1}), \quad (\text{B.11})$$

$$j = \frac{\tilde{\lambda}}{l}f + \lambda^\phi \partial_\phi f + \frac{l}{r}(K + \frac{1}{l})f = O(r^{-3}). \quad (\text{B.12})$$

The four functions  $f, \chi, \psi$  and  $Y$  are independent of  $r$ . A useful result is:

$$\frac{1}{r}(\partial_r - \frac{1}{r})\kappa = \frac{l^2}{r^3}\chi + \frac{\tilde{\lambda}}{l}f + \lambda^\phi \partial_\phi f + \frac{l}{r}(K + \frac{1}{l})f \quad (\text{B.13})$$

Let's assume that our improper gauge parameters are independent of the dynamical variables. The bracket of two gauge transformations is

$$[\xi_1, \xi_2]^\mu \approx [\xi_1, \xi_2]_{SD}^\mu - \delta_1^z \xi_2^\mu + \delta_2^z \xi_1^\mu. \quad (\text{B.14})$$

where the variation  $\delta^z$  hits the dynamical variables only and  $\xi^\mu = (\xi, \xi^r, \xi^\phi)$ . Long computations lead to

$$\hat{f} = Y_1 \partial_\phi f_2 + \psi_1 f_2 - (1 \leftrightarrow 2) \quad (\text{B.15})$$

$$\hat{\chi} = Y_1 \partial_\phi \chi_2 - \psi_1 \chi_2 - D_\phi(f_2 \bar{\gamma}^{-1} \partial_\phi \psi_1) - \frac{P}{2\sqrt{\gamma}} f_1 \chi_2 - (1 \leftrightarrow 2) \quad (\text{B.16})$$

$$\hat{Y} = Y_1 \partial_\phi Y_2 + \bar{\gamma}^{-1} f_1 \partial_\phi f_2 - (1 \leftrightarrow 2) \quad (\text{B.17})$$

$$\hat{\psi} = Y_1 \partial_\phi \psi_2 + f_1 \chi_2 - (1 \leftrightarrow 2), \quad (\text{B.18})$$

where the hatted quantities parametrize the resulting transformation  $[\xi_1, \xi_2]$ .

## C Fefferman-Graham gauge fixing

Some computations are a lot easier when the gauge is fixed. In this work, we are using the Fefferman-Graham choice [16, 17, 18, 19, 20, 21]. However, due to the relaxed asymptotics and the fact that we are working in the hamiltonian framework, it is not clear if this gauge can be reached with a proper gauge transformation.

In coordinates  $x^\mu = (r, t, \phi)$  the FG gauge is given by  $g_{rr} = \frac{l^2}{r^2}$  and  $g_{rt} = 0 = g_{r\phi}$ . On the canonical variables, these conditions become

$$g_{rr} = \frac{l^2}{r^2}, \quad g_{r\phi} = 0, \quad \pi^{\phi\phi} = 0. \quad (\text{C.1})$$

The question now is how can we send a configuration satisfying our relaxed boundary conditions (2.4) and (2.5) onto this gauge-fixed surface?

Let's introduce auxiliary quantities  $\tilde{N}, \tilde{N}^i$  satisfying

$$\tilde{N} = \frac{r}{l} + O(r^{-3}), \quad \tilde{N}^r = O(r^{-1}), \quad \tilde{N}^\phi = O(r^{-2}). \quad (\text{C.2})$$



The exact value of these fields is not important. Using these to generate a time evolution along  $s$ , we can build an auxiliary 3 dimensional metric with coordinates  $x^\mu = (s, r, \phi)$ :

$$\tilde{g}_{ss} = -\tilde{N}^2 + \tilde{N}^i g_{ij} \tilde{N}^j, \quad \tilde{g}_{si} = g_{ij} \tilde{N}^j, \quad \tilde{g}_{ij} = g_{ij}, \quad (\text{C.3})$$

$$\partial_s g_{ij} = \left\{ g_{ij}, \int (\tilde{N} \mathcal{R} + \tilde{N}_i \mathcal{R}_i) \right\}, \quad (\text{C.4})$$

$$\partial_s \pi^{ij} = \left\{ \pi^{ij}, \int (\tilde{N} \mathcal{R} + \tilde{N}_i \mathcal{R}_i) \right\}. \quad (\text{C.5})$$

Because the lagrange multipliers we chose preserve the asymptotic behavior of the canonical fields under "time" evolution, the auxiliary metric takes the form

$$\tilde{g}_{rr} = \frac{l^2}{r^2} + O(r^{-4}), \quad \tilde{g}_{rs} = O(r^{-3}), \quad \tilde{g}_{r\phi} = O(r^{-1}), \quad (\text{C.6})$$

$$\tilde{g}_{ss} = -\frac{r^2}{l^2} + O(r^{-2}), \quad \tilde{g}_{s\phi} = O(1), \quad \tilde{g}_{\phi\phi} = r^2 \bar{\gamma} + O(1). \quad (\text{C.7})$$

This can be put into the Fefferman-Graham form using a change of coordinates of the form

$$r = r' + O(r'^{-1}), \quad (\text{C.8})$$

$$s = s' + O(r'^{-4}), \quad (\text{C.9})$$

$$\phi = \phi' + O(r'^{-2}). \quad (\text{C.10})$$

This transformation is the exponential of the transformation generated by a vector with the following asymptotic behavior

$${}^{(3)}\xi^r = O(r^{-1}), \quad {}^{(3)}\xi^s = O(r^{-4}), \quad {}^{(3)}\xi^\phi = O(r^{-2}), \quad (\text{C.11})$$

which, brought back to the hamiltonian formalism using the lagrange multipliers  $\tilde{N}$  and  $\tilde{N}^i$ , is a proper gauge transformation:

$$\xi = O(r^{-3}), \quad \xi^r = O(r^{-1}), \quad \xi^\phi = O(r^{-2}). \quad (\text{C.12})$$

## References

- [1] S. Deser and R. Jackiw, "Three-dimensional cosmological gravity: Dynamics of constant curvature," *Annals Phys.* **153** (1984) 405–416.
- [2] M. Banados, C. Teitelboim, and J. Zanelli, "Black hole in three-dimensional spacetime," *Physical Review Letters* **69** (1992), no. 13, 1849.
- [3] M. Banados, M. Henneaux, C. Teitelboim, and J. Zanelli, "Geometry of the 2+1 black hole," *Physical Review D* **48** (1993), no. 4, 1506.

- [4] J. D. Brown and M. Henneaux, “Central charges in the canonical realization of asymptotic symmetries: an example from three dimensional gravity,” *Communications in Mathematical Physics* **104** (1986), no. 2, 207–226.
- [5] A. Strominger, “Black hole entropy from near-horizon microstates,” *Journal of High Energy Physics* **1998** (1998), no. 02, 009.
- [6] A. P. Porfyriadis and F. Wilczek, “Effective action, boundary conditions, and virasoro algebra for  $ads_3$ ,” *arXiv preprint arXiv:1007.1031* (2010).
- [7] C. Troessaert, “Enhanced asymptotic symmetry algebra of  $ads_3$ ,” *Journal of High Energy Physics* **2013** (2013), no. 8, 1–17.
- [8] G. Compère, W. Song, and A. Strominger, “New boundary conditions for  $ads_3$ ,” *Journal of High Energy Physics* **2013** (2013), no. 5, 1–26.
- [9] S. G. Avery, R. R. Poojary, and N. V. Suryanarayana, “An  $sl(2, \mathbb{R})$  current algebra from  $ads_3$  gravity,” *Journal of High Energy Physics* **2014** (2014), no. 1, 1–9.
- [10] O. Coussaert, M. Henneaux, and P. van Driel, “The asymptotic dynamics of three-dimensional einstein gravity with a negative cosmological constant,” *Classical and Quantum Gravity* **12** (1995), no. 12, 2961.
- [11] M. Henneaux, L. Maoz, and A. Schwimmer, “Asymptotic dynamics and asymptotic symmetries of three-dimensional extended  $ads$  supergravity,” *Annals of Physics* **282** (2000), no. 1, 31–66.
- [12] M. Rooman and P. Spindel, “Holonomies, anomalies and the fefferman–graham ambiguity in  $ads_3$  gravity,” *Nuclear physics B* **594** (2001), no. 1, 329–353.
- [13] G. Barnich and H. A. González, “Dual dynamics of three dimensional asymptotically flat einstein gravity at null infinity,” *Journal of High Energy Physics* **2013** (2013), no. 5, 1–27.
- [14] C. Troessaert, “Canonical structure of field theories with boundaries and applications to gauge theories,” *arXiv preprint arXiv:1312.6427* (2013).
- [15] L. Apolo and M. Porrati, “Free boundary conditions and the  $ads_3/cft_2$  correspondence,” *Journal of High Energy Physics* **2014** (2014), no. 3, 1–27.
- [16] C. Fefferman and C. Graham, “Conformal invariants,” *Elie Cartan et les Mathématiques d’aujourd’hui* (1985) 95–116.
- [17] C. Graham and J. Lee, “Einstein metrics with prescribed conformal infinity on the ball,” *Adv. Math.* **87** (1991) 186–225.

- [18] K. Skenderis and S. N. Solodukhin, “Quantum effective action from the ads/cft correspondence,” *Physics Letters B* **472** (2000), no. 3, 316–322.
- [19] C. R. Graham, “Volume and area renormalizations for conformally compact einstein metrics,” *arXiv preprint math/9909042* (1999).
- [20] M. R. Rooman and P. Spindel, “Aspects of  $(2+1)$  dimensional gravity: Ads3 asymptotic dynamics in the framework of fefferman-graham-lee theorems,” *arXiv preprint hep-th/9911142* (1999).
- [21] K. Bautier, F. Englert, M. Rooman, and P. Spindel, “The fefferman-graham ambiguity and ads black holes,” *Physics Letters B* **479** (2000), no. 1, 291–298.